Observer Design for a Class of Parabolic PDE Via Sliding Modes and Backstepping

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***Resumen*— Observation problem for systems governed by Partial Differential Equations (PDE) has been a research field of its own for a long time. In this paper it is presented an observer design for a class or parabolic PDE's using sliding modes theory and bacstepping-like procedure in order to achieve exponential convergence. A Volterra-like integral transformation is used to change the coordinates of the error dynamics into exponentially stable target systems using the backstepping-like procedure. This gives as a result the output injection functions of the observer which are obtained by solving a hyperbolic PDE system. Sliding modes are used to find an explicit solution to the hyperbolic PDE system and to make the observer gains to be discontinuous which have well known advantages. Theoretical results were proved using the Lyapunov theory. A numerical example demonstrates the proposed method effectiveness.**

I. INTRODUCTION

I-A. PDE Observers

There exist a great deal of systems which dynamical behavior are described by Partial Differential Equations (PDE), (Schiesser y Griffiths, 2009). In recent years, this field has broadened considerably as more realistic models have been introduced and investigated in different areas such as thermodynamics, elastic structures, fluid dynamics and biological systems, to name some few (Imanuvilov y Bing-Yu, 2005), (Mondaini y Pardalos, 2008).

In spite of the fact that optimization and control of systems governed by partial differential equations and more recently by variational inequalities is a very active field of research (Kunisch y Tröltzsch, 2007), no much have been developed for observer design. Two main results of this field are described in (Krstic y Smyshlyaev, 2005) and (Orlov, 2009).

The first one introduces an observer design for linear parabolic PDE. It involves a linear Volterra transformation of the observer error system into a target system which is exponentially stable. If the kernel transformation, that satisfies a linear hyperbolic PDE, is causal, the output error injection functions in the structure of the observer proposed there are then given and depends on this kernel. Some application of this obsever are presented in (Krstic y Smyshlyaev, 2007), (Vazquez y Krstic, 2005).

On the other hand, the second result describes how an infinite-dimensional Luenberger state observer, which utilizes a finite number of measurements, is constructed to provide estimates of the state of the infinite-dimensional system. Furthermore, the infinite-dimensional Luenberger state observer can be replaced by its sliding mode counterpart as shown in (Orlov y Dochain, 2002). This takes advantage of the backstepping method for PDE and the sliding mode theory for infinite dimensional systems. The next subsections describes the basis of these couple of techniques.

I-B. Backstepping Theory for PDE

Backstepping was developed to stabilize dynamical systems and is particularly successful in the area of nonlinear control, (Krstic y Kokotovic, 1995). The basic idea of backstepping technique for infinite dimensional systems was developed in (Balogh y Krstic, 2004), where stability properties of a class of LTV difference equations on an infinite-dimensional state space were studied. This LTV difference equations were obtained because of the original systems discretization (which is wanted to be transformed), target system (in which the original system will be transformed) and the proper integral transformation which results in recursive relationships backstepping-like. Stabilization of the original system is guaranteed if the kernel of the transformation is causal. The extension to this idea into systems governed by PDE for state estimation was proposed in (Krstic y Smyshlyaev, 2005). This causality of a backstepping-like transformation, applied to the error system leads to an exponentially stable target system, is needed to obtain the output injection functions of the proposed observer.

I-C. Sliding Modes

Sliding mode control applied into systems modelled by ordinary differential equations have guaranteed state estimation with a certain degree of robustness (Benallegue y L, 2007), (Davila y Poznyak, 2006), (Davila y Levant, 2005) and (Fridma y Yan, 2008). Extending this powerful method into distributed parameter systems, more specific, into systems modeled by PDE have been recently researched (Orlov, 2009). However, finite time convergence that is possible, under some considerations, in finite-dimensional systems, is only possible in plants which dynamics could be represented by PDE if measures can be done in whole space for all time. In cases where measure is only possible in the boudary, it is expected, at most, finite time convergence only in the measure point, but, exponential convergence at the rest of the space.

I-D. Main contribution of this paper

In this paper it is presented the combination of sliding modes approach and the backstepping-like transformation applied into systems described by a class of parabolic partial differential equations in order to make an observer design for parabolic PDE. This paper is organized as follows. Section II first describes the class of PDE and boundary conditions in which a plant must be described in order to be capable to be observed. Secondly, it is presented a non linear transformation which represents this class of PDE in an particular form that is wanted to directly apply the observer design.

Section III introduces the proper structure of the observer. This observer is described by almost the same equation than the plant just by adding two new error correction functions which injects the measured output that is only aviable at the boundary $x = 0$. Then the error dynamics are obtained and it is introduced an Volterra-like integral transformation with the aim of transform the error dynamics into an exponentially stable target system. The stability of the target system is proved using Lyapunov theory which has had recently advances in sliding modes theory (Dávila y Fridman, 2009). The theorem presented next summarizes the conditions needed to demonstrate the capability of apply the Volterra-Like integral transformation.

Proof of the theorem is presented in section IV where the explicit conditions are detailed. The kernel behavior modelled by hyperbolic PDEs and the output injection functions in the observer structure are obtained. This section ends with another theorem necessary to verify the Volterra-like transformation causality.

Proof of this new theorem is presented in section V. The existence of another kernel modelled by another hyperbolic PDEs shows that the integral transformation is invertible and then causal.

Finally, section VI presents the conclusions.

II. Class of Parabolic PDE

Lets consider the following class of parabolic PDE:

$$\begin{array}{c} u_t(x,t) = a u_{xx}(x,t) + b(x) u_x(x,t) \\ + c(x) u(x,t) + g(x) u(0,t) \end{array} \quad (1)$$

where

$$u_t(x,t) := \frac{\partial u(x,t)}{\partial t}$$
$$u_x(x,t) := \frac{\partial u(x,t)}{\partial x} \qquad u_{xx}(x,t) := \frac{\partial^2 u(x,t)}{\partial x^2}$$

The PDE domain is $x \in [0,1]$, $t > 0$ and its boundary conditions are

$$\begin{array}{c} u_x(0,t) = q u(0,t) \\ u(1,t) = U \end{array} \quad (2)$$

Parameters and nonlinear functions involved in (1) fulfill the assumptions given in

$$a > 0, \; q \in \mathbb{R} \qquad b(\cdot), c(\cdot) \in C^1[0,1]$$
$$\int c(x)\, dx|_{x=1} = -\mu \qquad f \in C^1\{[0,1] \times [0,1]\}$$

In (Balogh y Krstic, 2004), it has been shown that without loss of generality, $b(x)$ can be set to zero since there exist a nonlinear transformation

$$u(x,t) \rightarrow u(x,t) e^{r(x)} \qquad r(x) := -\frac{1}{2a}\int_0^x b(y)\, dy$$

such that (1) may be converted into

$$u_t(x,t) = \bar{a} u_{xx}(x,t) + \bar{c}(x) u(x,t) + \bar{g}(x) u(0,t)$$

where

$$\bar{a} := a$$
$$\bar{c}(x) := -b_x(x) - \frac{1}{4a} b^2(x) + c(x)$$
$$\bar{g}(x) := \frac{g(x) e^{r(0)}}{e^{r(x)}}; \; r(x) > -\infty$$

Boundary conditions described in (2) indicate that PDE (1) is actuated at $x = 1$ (nonetheless if the actuation is Dirichlet or Neumman) by an input $U(t)$ that can be selected as a function of time or a feedback design.

III. Sliding Mode Observer

The observer is in the anticollocated setup according with (Krstic y Smyshlyaev, 2008) and it satisfies the following parabolic PDE

$$\begin{array}{c} \hat{u}_t(x,t) = \bar{a}\hat{u}_{xx}(x,t) + g(x) u(0,t) + \bar{c}(x)\hat{u}(x,t) \\ + K_0(x)\,\mathrm{sign}(u(0,t) - \hat{u}(0,t)) \\ + K_1(x)(u(0,t) - \hat{u}(0,t)) \\ \\ \hat{u}_x(0,t) = q u(0,t) + K_2(t)\mathrm{sign}[u(0,t) - \hat{u}(0,t)] \\ + K_3 \tilde{u}(0,t) \\ \\ \hat{u}(1,t) = U(t) \end{array} \quad (3)$$

The output's error injection functions $K_0(x)$, $K_1(x)$, K_2 and K_3 should be designed using the backsteeping procedure proposed in (Krstic y Smyshlyaev, 2008), (Krstic y Smyshlyaev, 2005). Following the above mentioned method, the output injection is applied in the boundary as well as in the whole spacial domain ($\forall x \in [0,1]$).The observation error is defined as follows

$$\tilde{u}(x,t) := u(x,t) - \hat{u}(x,t)$$

and its dynamics satisfies the following parabolic PDE

$$\begin{array}{c} \tilde{u}_t(x,t) = a\tilde{u}_{xx}(x,t) + c(x)\tilde{u}(x,t) - K_0(x)\,\mathrm{sign}(\tilde{u}(0,t)) \\ - K_1(x)\tilde{u}(0,t) \\ \\ \tilde{u}_x(0,t) = -K_2 \mathrm{sign}(\tilde{u}(0,t)) - K_3\tilde{u}(0,t) \\ \tilde{u}(1,t) = 0 \end{array} \quad (4)$$

Gains matrices $K_0(x)$, $K_1(x)$, K_2 and K_3 should be selected to stabilize the error dynamics (4). The problem will be solved using the backstepping-technique described in (Cochran y Krstic, 2006), (Krstic y Smyshlyaev, 2005) and (Xu y Krstic, 2007). This method uses a integral transformation, that is a backstepping-like coordinate transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \int_0^x K(x,y)\,\tilde{u}(y,t)\,dy \quad (5)$$

The application of such transformation makes the error system (4) be transformed in an exponentially stable (for $\tilde{\mu} \geq 0$) system:

$$\begin{array}{c} \tilde{w}_t(x,t) = \bar{a}\tilde{w}_{xx}(x,t) - \tilde{\mu}\tilde{w}(x,t) \\ +\tilde{K}_0(x)\mathrm{sign}(\tilde{w}(0,t)) + \tilde{K}_1(x)\tilde{w}(0,t) \\ \tilde{w}_x(0,t) = -q_1\tilde{w}(0,t) - q_2\mathrm{sign}(\tilde{w}(0,t)) \\ \tilde{w}(1,t) = 0 \end{array} \quad (6)$$

It is easy to proof that the target system (6) is exponentially stable since there exist a Lyapunov function which demonstrate, under the assumptions given before, that (6) converge exponentially to $\tilde{w} = 0$ for every x and for every initial condition.

Therefore the main result regarding to this anti-collocated observer is described by the following theorem.

Teorema 1: Let the system be in the form (1) with the condition mentiond in (2). Let the designed observer be (3). Then the error dynamic is given by (4). The integral transform shown in (5) transforms the error dynamic into the exponentially stable system desbribed in (6) which ensures that the observer (3) converge exponentially.into the system (1).

The proof of this theorem is shown in next section.

IV. Direct Transformation. Proof of the Theorem

Let the target system (6) and the integral transformation (5). Differentiating (5) with respect to time, it is obtained:

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - \int_0^x K(x,y)\,\tilde{u}_t(y,t)\,dy$$

Substituting $\tilde{u}_t(x,t)$ one gets

$$\begin{array}{c} \tilde{w}_t(x,t) = a\tilde{u}_{xx}(x,t) + c(x)\,\tilde{u}(x,t) \\ -K_0(x)\,\mathrm{sign}(\tilde{u}(0,t)) \\ +\int_0^x c(y)\,\tilde{u}(y,t)\,K(x,y)\,dy \\ -\int_0^x K_0(y)\,\mathrm{sign}(\tilde{u}(0,t))\,K(x,y)\,dy \\ -\int_0^x K_1(y)\,\tilde{u}(0,t)\,K(x,y)\,dy + \int_0^x a\tilde{u}(y,t)\,K(x,y)\,dy \\ +a\tilde{u}_x(x,t)\,K(x,x) - a\tilde{u}_x(0,t)\,K(x,0) \\ -a\tilde{u}(x,t)\,K_x(x,x) + a\tilde{u}(0,t)\,K_x(x,0) \end{array}$$

Now, differentiating twice with respect to x,

$$\begin{array}{c} \tilde{w}_{xx}(x,t) = \tilde{u}_{xx}(x,t) - \left(\frac{d}{dx}K(x,x)\right)\tilde{u}(x,t) \\ -K(x,x)\,\tilde{u}_x(x,t) - K_x(x,x)\,\tilde{u}(x,t) \\ -\int_0^x K_{xx}(x,y)\,\tilde{u}(y,t)\,dy \end{array}$$

Considering the transformation (5) and simplifying we get

$$\begin{array}{c} \left[\tilde{\mu} + c(x) - 2aK_x(x,x) - a\left(\frac{d}{dx}K(x,x)\right)\right]\tilde{u}(x,t) \\ +\int_0^x \left[aK_{yy}(x,y) + \tilde{\mu}K(x,y)\right]\tilde{u}(y,t)\,dy \\ +\int_0^x \left[c(y)K(x,y) - aK_{xx}(x,y)\right]\tilde{u}(y,t)\,dy \\ -\tilde{K}_0(x)\mathrm{sign}(\tilde{u}(0,t)) - K_0(x)\,\mathrm{sign}(\tilde{u}(0,t)) \\ -K_1(x)\tilde{u}(0,t) - \tilde{K}_1(x)\tilde{u}(0,t) \\ -\int_0^x K_0(y)\,\mathrm{sign}(\tilde{u}(0,t))\,K(x,y)\,dy \\ -\int_0^x K_1(y)\,\tilde{u}(0,t)\,K(x,y)\,dy \\ -a\tilde{u}_x(0,t)\,K(x,0) + a\tilde{u}(0,t)\,K_x(x,0) = 0 \end{array}$$

This equation can be satisfied in the following equations are fulfilled

$$\begin{array}{c} \tilde{\mu} + c(x) - 2aK_x(x,x) - a\left(\frac{d}{dx}K(x,x)\right) = 0 \\ -aK_{xx}(x,y) + \tilde{\mu}K(x,y) + c(y)K(x,y) + aK_{yy}(x,y) = 0 \\ K(x,0) = 0, \qquad K(1,y) = 0 \end{array} \quad (7)$$

Integrating along x the first equation of (7) we get

$$\begin{array}{c} \tilde{\mu}x + \int c(x)\,dx - 2aK(x,x) - aK(x,x) = 0 \\ K(x,x) = \frac{\tilde{\mu}x + \int c(x)dx}{3a} \end{array}$$

Then the kernel dynamics are described in (7) and the output injection functions are:

$$K_0(x) = \int_0^x K_0(y)\,K(x,y)\,dy - \tilde{K}_0(x)$$

$$K_1(x) = \int_0^x K_1(y)\,K(x,y)\,dy - \tilde{K}_1(x)$$

considering the boundary condition we get

$$\begin{array}{c} \tilde{w}_x(x,t) = \tilde{u}_x(x,t) - K(x,x)\,\tilde{u}(x,t) \\ -\int_0^x K_x(x,y)\,\tilde{u}(y,t)\,dy \end{array}$$

evaluating in $x=0$ we obtain

$$\tilde{w}_x(0,t)=\tilde{u}_x(0,t) \\ =-K_2\text{sign}(\tilde{u}(0,t))-K_3\tilde{u}(0,t)$$

y por lo tanto

$$K_3=q_1 \qquad K_2=q_2$$

In order to complete the proof, the following theorem is needed.

Teorema 2: Let the target system be (6). Let the error dynamics be described by (4). If there exist a new integral transformation involving a new kernel dynamics which can transform the target system (6) into the error dynamics (4) and if the new kernel dynamics are well defined, then the integral transformation (5) is said to be causal and it is possible to be applied into observer designed (3)

The proof of this theorem is shown in next section.

V. Causality Proof. Inverse Transformation

In order to complete the observer design, it is necessary to establish that the target system (6) implies stability to the closed loop system (4) with (5). In other words it is considered that the integral transformation (5) is invertible. To achieve this, a new integral transformation is proposed:

$$\tilde{u}(x,t)=\tilde{w}(x,t)+\int_0^x L(x,y)\,\tilde{w}(y,t)\,dy \qquad (8)$$

where $L(x,y)$ is the kernel of this new transformation. The process that was carried out to obtain this new transformation depends on the use of the original error system (4) and the target system (6). Differentiating (8) with respect to time it is obtained:

$$\tilde{u}_t(x,t)=\tilde{w}_t(x,t)+\int_0^x L(x,y)\,\tilde{w}_t(y,t)\,dy$$

Using $\tilde{w}_t(x,t)$, we get

$$\tilde{u}_t(x,t)=\bar{a}\tilde{w}_{xx}(x,t)-\tilde{\mu}\tilde{w}(x,t) \\ +\tilde{K}_0(x)\text{sign}(\tilde{w}(0,t))+\tilde{K}_1(x)\tilde{w}(0,t) \\ -\int_0^x \tilde{\mu}\tilde{w}(y,t)\,L(x,y)\,dy \\ +\int_0^x \tilde{K}_0(y)\text{sign}(\tilde{w}(0,t))L(x,y)\,dy \\ +\int_0^x \tilde{K}_1(y)\tilde{w}(0,t)L(x,y)\,dy \\ +\bar{a}\tilde{w}_x(x,t)\,L(x,x)-\bar{a}\tilde{w}_x(0,t)\,L(x,0) \\ -\bar{a}\tilde{w}(x,t)\,L_x(x,x)+\bar{a}\tilde{w}(0,t)\,L_x(x,0) \\ +\int_0^x \bar{a}\tilde{w}(y,t)\,L_{yy}(x,y)\,dy$$

Following with (8), and differentiating with respect to x twice we get

$$\tilde{u}_{xx}(x,t)=\tilde{w}_{xx}(x,t)+\tilde{w}_x(x,t)\,L(x,x)+\tilde{w}(x,t)\,\tfrac{dL(x,x)}{dx} \\ +L_x(x,x)\,\tilde{w}(x,t)+\int_0^x L_{xx}(x,y)\,\tilde{w}(y,t)\,dy$$

Substituting $\tilde{u}_t(x,t)$ and $\tilde{u}_{xx}(x,t)$ in (4) we get

$$\left[-c(x)-\tilde{\mu}-2\bar{a}L_x(x,x)-a\tfrac{dL(x,x)}{dx}\right]\tilde{w}(x,t) \\ +\int_0^x\left[-c(x)\,L(x,y)-\tilde{\mu}L(x,y)\right]\tilde{w}(y,t)\,dy \\ +\int_0^x\left[\bar{a}L_{yy}(x,y)-\bar{a}L_{xx}(x,y)\right]\tilde{w}(y,t)\,dy \\ +K_0(x)\,\text{sign}(\tilde{u}(0,t))+K_1(x)\tilde{u}(0,t) \\ +\tilde{K}_0(x)\text{sign}(\tilde{w}(0,t))+\tilde{K}_1(x)\tilde{w}(0,t) \\ +\int_0^x \tilde{K}_0(y)\text{sign}(\tilde{w}(0,t))L(x,y)\,dy \\ +\int_0^x \tilde{K}_1(y)\tilde{w}(0,t)L(x,y)\,dy \\ -\bar{a}\tilde{w}_x(0,t)\,L(x,0)+\bar{a}\tilde{w}(0,t)\,L_x(x,0)=0$$

In order to conclude this proof, the following conditions must be satisfied

$$-c(x)-\tilde{\mu}-2\bar{a}L_x(x,x)-a\tfrac{dL(x,x)}{dx}=0 \\ -c(x)\,L(x,y)-\tilde{\mu}L(x,y)+\bar{a}L_{yy}(x,y)-\bar{a}L_{xx}(x,y)=0 \\ L(x,0)=0 \\ \tilde{K}_1(x)=\int_0^x \tilde{K}_1(y)L(x,y)\,dy-K_1(x) \\ \tilde{K}_2(x)=\int_0^x \tilde{K}_2(y)L(x,y)\,dy-K_2(x) \qquad (9)$$

The kernel $L(x,y)$ satisfies the hyperbolic PDE described in (9).

VI. NUMERICAL SIMULATION RESULTS

The system considered in this example is described by the following PDE

$$u_t(x,t)=0{,}01u_{xx}(x,t) \\ -(0{,}2\sin(\pi t)+0{,}01\sin(2\pi t)+0{,}015\sin(0{,}5\pi t))\,u(x,t)$$

The data aviable is according with (2) in the boundary $u(0,t)$ and the control is in the boundary $u(1,t)=\sin(1{,}7\pi t)+5$.

The dynamics of this system is shown in (1). Where the initial conditions are chosen arbitrarily

The kernel behavior is shown in figure (2). It is clear that the kernel satisfies conditions mentioned in (7). Oscillation

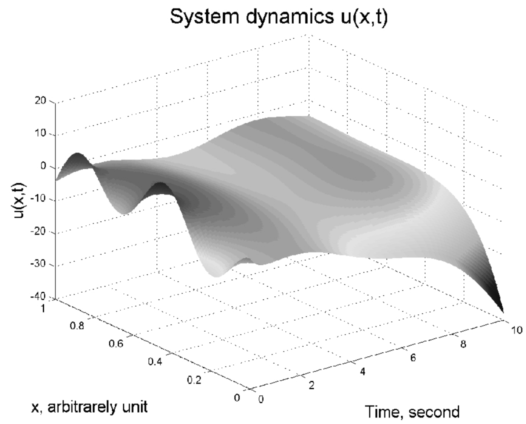

Figura 1. Fig. 1. Here is shown the system dynamical behabior without input. The space in x belongs to $[0\,,\,1]$.

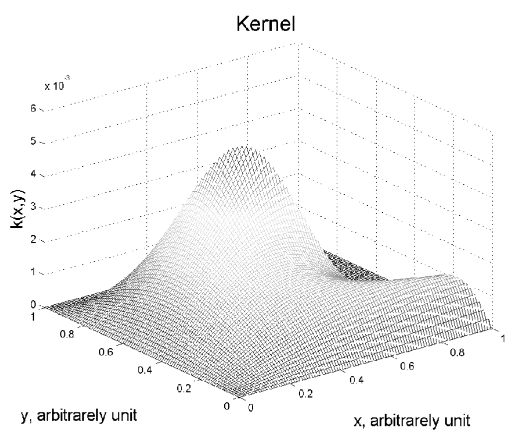

Figura 2. Fig. 2. Evolution of the kernel behavior along x and y. Oscilations depends on the selected values of a, μ and $c(x)$.

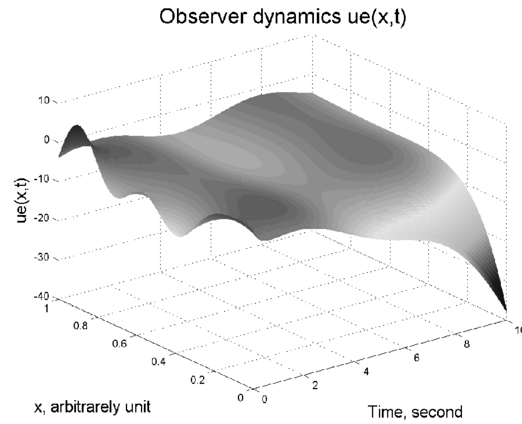

Figura 3. Fig. 3. Observer dynamics shown here demonstrate the capability of the observer to reach the system dynamics starting with different initial conditions.

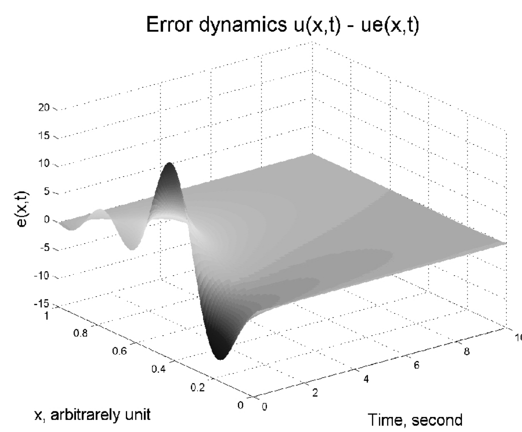

Figura 4. Fig. 4. Error dynamics goes to zero exponentially in time for every value of x with different decreasing rates.

along x are given by different values in parameters a and $c(x)$.

Once the kernel was numerically calculated, the output injection functions can be numerically calculated too and the observer is able to be computed. In figure (3) is shown the dynamical behavior of the observer with different initial conditions than those applied to the plant.

As it is not good to plot system and observer in the same chart, one way to show how does the observer converge to the system is by plotting the behavior of the error along time. The figure (4) shows the dynamical behavior of error described in (4).

As it was expected, in the boundary $x = 0$ the observer converge in finite time. However, in other values of x the convergence is exponential. This can be seen in figure (5) where arbitrarily values where chosen to show how is the evolution of $u(x,t)$ along time when $x = 0$, $x = 0{,}25$, $x = 0{,}75$ and $x = 1$.

VII. CONCLUSIONS

In this paper was presented an observer design for a class or parabolic PDEs using sliding modes theory and backstepping-like procedure obtaining exponential convergence in every x except in $x = 0$ where finite time convergence was achieved. A Volterra-like integral transformation was used to make coordinates changing from error dynamics to exponentially stable target systems. Correction functions in the observer structure are obtained by solving a

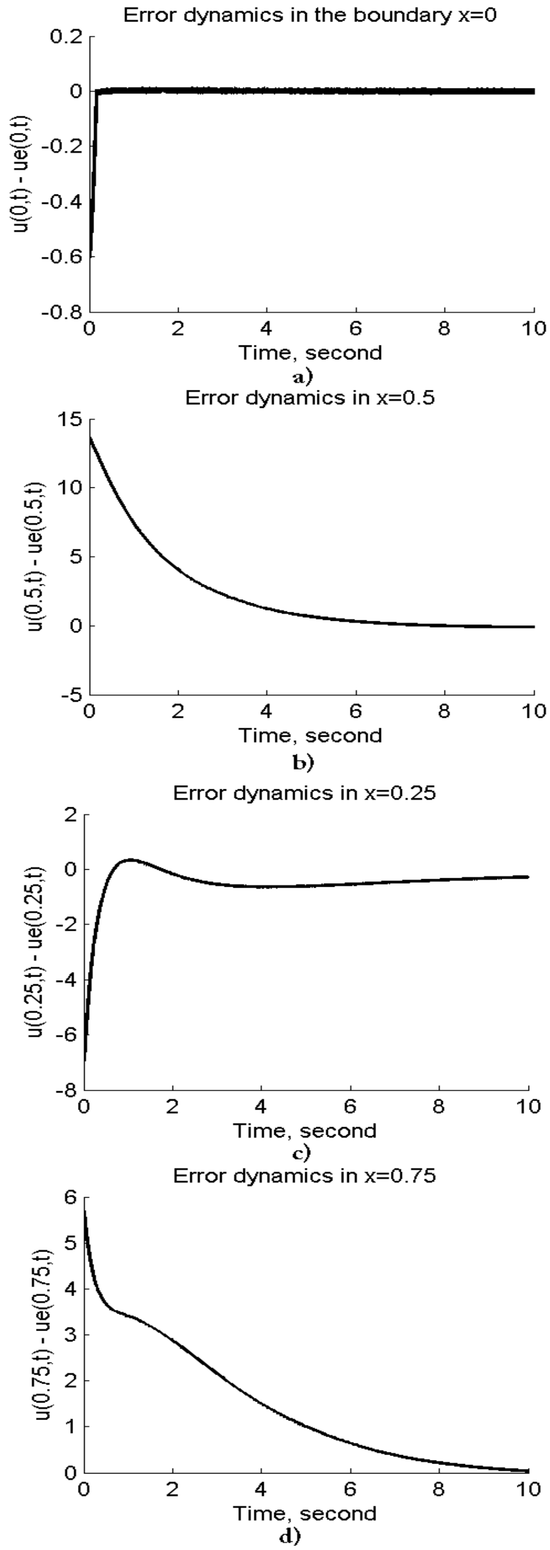

Figura 5.   Fig. 5. Convergence in differente spatial points. a) System dynamics in $x = 0$ where finite time convergence was obtained. b) Exponetial convergence was obtained in $x = 0{,}25$. c) Similar results in $x = 0{,}5$ and in $x = 0{,}75$ shown in d).

kernel modelled by hyperbolic PDEs. The causality of this transformation was proved verifying that the invertibility existence which depends on the existence solution of another kernel in hyperbolic PDE. Sliding modes were used to find an explicit solution to the hyperbolic PDE. Stability was proved using Lyapunov theory. Numerical results demonstrate the effectiveness of applying this observer.

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